

Some bounds on the number of colors in interval and cyclic interval edge colorings of graphs

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November 22, 2016

Abstract. An *interval t -coloring* of a multigraph G is a proper edge coloring with colors $1, \dots, t$ such that the colors on the edges incident to every vertex of G are colored by consecutive colors. A *cyclic interval t -coloring* of a multigraph G is a proper edge coloring with colors $1, \dots, t$ such that the colors on the edges incident to every vertex of G are colored by consecutive colors, under the condition that color 1 is considered as consecutive to color t . Denote by $w(G)$ ($w_c(G)$) and $W(G)$ ($W_c(G)$) the minimum and maximum number of colors in a (cyclic) interval coloring of a multigraph G , respectively. We present some new sharp bounds on $w(G)$ and $W(G)$ for multigraphs G satisfying various conditions. In particular, we show that if G is a 2-connected multigraph with an interval coloring, then $W(G) \leq 1 + \left\lfloor \frac{|V(G)|}{2} \right\rfloor (\Delta(G) - 1)$. We also give several results towards the general conjecture that $W_c(G) \leq |V(G)|$ for any triangle-free graph G with a cyclic interval coloring; we establish that approximate versions of this conjecture hold for several families of graphs, and we prove that the conjecture is true for graphs with maximum degree at most 4.

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1 Introduction

In this paper we consider graphs which are finite, undirected, and have no loops or multiple edges, and multigraphs which may contain multiple edges but no loops. We denote by $V(G)$ and $E(G)$ the sets of vertices and edges of a multigraph G , respectively, and by $\Delta(G)$ and $\delta(G)$ the maximum and minimum degrees of vertices in G , respectively. The terms and concepts that we do not define here can be found in [2, 26].

An *interval coloring* (or *consecutive coloring*) of a multigraph G is a proper coloring of the edges by positive integers such that the colors on the edges incident to any vertex of G form an interval of integers, and no color class is empty. The notion of interval colorings was introduced by Asratian and Kamalian [4] (available in English as [5]), motivated by the problem of finding compact school timetables, that is, timetables such that the lectures of each teacher and each class are scheduled at consecutive periods. We denote by \mathfrak{N} the set of all interval colorable multigraphs.

All regular bipartite graphs have interval colorings, since they decompose into perfect matchings. Not every graph has an interval coloring, since a graph G with an interval coloring must have a proper $\Delta(G)$ -edge-coloring [4]. Sevastjanov [24] proved that determining whether a bipartite graph has an interval coloring is \mathcal{NP} -complete. Nevertheless, trees [15], regular and complete bipartite graphs [15, 16, 12], grids [11], and outerplanar bipartite graphs [10, 6] all have interval colorings. Moreover, some families of (a, b) -biregular graphs have been proved to admit interval colorings [14, 12, 8, 3, 23, 27], where a bipartite graph is (a, b) -biregular if all vertices in one part have degree a and all vertices in the other part have degree b .

A *cyclic interval t -coloring* of a multigraph G is a proper t -edge-coloring α such that for every vertex v of G the colors on the edges incident to v either form an interval of integers or the set $\{1, \dots, t\} \setminus \{\alpha(e) : e \text{ is incident to } v\}$ is an interval of integers, and no color class is empty. This notion was introduced by de Werra and Solot [25] motivated by scheduling problems arising in flexible manufacturing systems, in particular the so-called *cylindrical open shop scheduling problem* [25]. We denote by \mathfrak{N}_c the set of all cyclically interval colorable multigraphs.

Cyclic interval colorings are studied in e.g. [18, 19, 22, 1]. In particular, the general question of determining whether a graph G has a cyclic interval coloring is \mathcal{NP} -complete [18] and there are concrete examples of connected graphs having no cyclic interval coloring [19, 22, 1]. Trivially, any multigraph with an interval coloring also has a cyclic interval coloring with $\Delta(G)$ colors, but the converse does not hold [19]. Graphs that have been proved to admit cyclic interval colorings (but not interval colorings) include all complete multipartite graphs [1], all Eulerian bipartite graphs of maximum degree at most 8 [1], and some families of (a, b) -biregular graphs [1, 7, 8].

In this paper we study upper and lower bounds on the number of colors in interval and cyclic interval colorings of various families of graphs. For a multigraph $G \in \mathfrak{N}$ ($G \in \mathfrak{N}_c$), we denote by $W(G)$ ($W_c(G)$) and $w(G)$ ($w_c(G)$) the maximum and the minimum number of colors in a (cyclic) interval coloring of a multigraph G , respectively. There are some previous results on these parameters in the literature. In particular, Asratian and Kamalian [4, 5] proved the fundamental result that if $G \in \mathfrak{N}$ is a triangle-free graph, then $W(G) \leq |V(G)| - 1$; this upper bound is sharp for e.g. complete bipartite graphs. Kamalian [16] proved that if

$G \in \mathfrak{N}$ has at least two vertices, then $W(G) \leq 2|V(G)| - 3$. In [9], it was noted that this upper bound can be slightly improved if the graph has at least three vertices. Petrosyan [20] proved that these upper bounds are asymptotically sharp by showing that for any $\epsilon > 0$, there is a connected interval colorable graph G satisfying $W(G) \geq (2 - \epsilon)|V(G)|$. Kamalian [15, 16] proved that the complete bipartite graph $K_{a,b}$ has an interval t -coloring if and only if $a + b - \gcd(a, b) \leq t \leq a + b - 1$, where $\gcd(a, b)$ is the greatest common divisor of a and b , and Petrosyan et al. [20, 21] showed that the n -dimensional hypercube Q_n has an interval t -coloring if and only if $n \leq t \leq \frac{n(n+1)}{2}$.

For cyclic interval colorings, Petrosyan and Mkhitarian [22] suggested the following:

Conjecture 1.1. (i) For any triangle-free graph $G \in \mathfrak{N}_c$, $W_c(G) \leq |V(G)|$.

(ii) For any graph $G \in \mathfrak{N}_c$ with at least two vertices, $W_c(G) \leq 2|V(G)| - 3$.

If true, then these upper bounds are sharp [22]. Using results on upper bounds for $W(G)$ [4, 16, 9], Petrosyan and Mkhitarian [22] proved that

- $W_c(G) \leq |V(G)| + \Delta(G) - 2$ if $G \in \mathfrak{N}_c$ is a triangle-free graph, and
- $W_c(G) \leq 2|V(G)| + \Delta(G) - 5$ for any graph $G \in \mathfrak{N}_c$ with at least three vertices.

In this paper we present some new general upper and lower bounds on the number of colors in interval and cyclic interval colorings of graphs.

For cyclic interval colorings, we prove several results related to Conjecture 1.1. In particular, we give improvements of the above-mentioned general upper bounds on $W_c(G)$ by Petrosyan and Mkhitarian, and we show that approximate versions of the conjecture hold for several families of graphs. We also prove the new general upper bound that for any triangle-free graph $G \in \mathfrak{N}_c$, $W_c(G) \leq \frac{\sqrt{3}+1}{2}(|V(G)| - 1)$, which is an improvement of the bound proved by Petrosyan et al. [22] for graphs with large maximum degree. These results are proved in Section 3, where we also prove that Conjecture 1.1 (i) is true for graphs with maximum degree at most 4.

For interval colorings, we prove that if G is a 2-connected multigraph and $G \in \mathfrak{N}$, then $W(G) \leq 1 + \left\lfloor \frac{|V(G)|}{2} \right\rfloor (\Delta(G) - 1)$. This result is proved in Section 4, where we also show that this upper bound is sharp, and give some related results. In Section 5, we obtain some lower bounds on $w(G)$ for multigraphs $G \in \mathfrak{N}$. We show that if G is an interval colorable multigraph, then $w(G) \geq \left\lceil \frac{|V(G)|}{2\alpha'(G)} \right\rceil \delta(G)$, where $\alpha'(G)$ is the size of a maximum matching in G . In particular, this implies that if G has no perfect matching and $G \in \mathfrak{N}$, then $w(G) \geq \max\{\Delta(G), 2\delta(G)\}$. Additionally, we prove that the same conclusion holds under the assumption that all vertex degrees in G are odd, $G \in \mathfrak{N}$ and $|E(G)| - \frac{|V(G)|}{2}$ is odd. All these lower bounds are sharp.

1.1 Notation and preliminary results

In this section we introduce some terminology and notation, and state some auxiliary results.

The set of neighbors of a vertex v in G is denoted by $N_G(v)$. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$ (or just $d(v)$), the maximum degree of vertices in G by $\Delta(G)$, the minimum degree of vertices in G by $\delta(G)$, and the average degree of G by $d(G)$.

A multigraph G is *even* (*odd*) if the degree of every vertex of G is even (odd). G is *Eulerian* if it has a closed trail containing every edge of G . For two distinct vertices u and v of a multigraph G , let $E(uv)$ denote the set of all edges of G joining u with v .

The diameter of G , i.e. the greatest distance between any pair of vertices in G , is denoted by $\text{diam}(G)$, and the circumference of G , i.e. the length of a longest cycle in G , is denoted by $c(G)$. We denote by $\alpha'(G)$ the size of a maximum matching in G , and by $\chi'(G)$ the chromatic index of G .

If α is a proper edge coloring of G and $v \in V(G)$, then $S_G(v, \alpha)$ (or $S(v, \alpha)$) denotes the set of colors appearing on edges incident to v . The smallest and largest colors of $S(v, \alpha)$ are denoted by $\underline{S}(v, \alpha)$ and $\overline{S}(v, \alpha)$, respectively.

We shall use the following theorem due to Asratian and Kamalian [4, 5].

Theorem 1.2. *If G is a triangle-free graph and $G \in \mathfrak{N}$, then*

$$W(G) \leq |V(G)| - 1.$$

In particular, from this result it follows that if G is a bipartite graph and $G \in \mathfrak{N}$, then $W(G) \leq |V(G)| - 1$.

For general graphs, Kamalian [16] proved the following.

Theorem 1.3. *If G is a graph with at least two vertices and $G \in \mathfrak{N}$, then*

$$W(G) \leq 2|V(G)| - 3.$$

Note that the upper bound in Theorem 1.3 is sharp for K_2 , but if $G \neq K_2$, then this upper bound can be improved.

Theorem 1.4. [9] *If G is a graph with at least three vertices and $G \in \mathfrak{N}$, then*

$$W(G) \leq 2|V(G)| - 4.$$

We shall also use the following result due to Asratian and Kamalian [4, 5].

Proposition 1.5. *If $G \in \mathfrak{N}$, then $\chi'(G) = \Delta(G)$. Moreover, if G is a regular multigraph, then $G \in \mathfrak{N}$ if and only if $\chi'(G) = \Delta(G)$.*

2 Cyclic interval colorings

In this section we prove several results related to Conjecture 1.1. Petrosyan and Mkhitaryan [22] proved the following general bounds.

Theorem 2.1. (i) *If G is a connected triangle-free graph with at least two vertices and $G \in \mathfrak{N}_c$, then $W_c(G) \leq |V(G)| + \Delta(G) - 2$.*

(ii) *If G is a connected graph with at least three vertices and $G \in \mathfrak{N}_c$, then $W_c(G) \leq 2|V(G)| + \Delta(G) - 5$.*

Part (i) of the above theorem implies that Conjecture 1.1 (i) holds for any graph with maximum degree at most 2. We give a slight improvement for graphs with maximum degree at least 3 of their general bound.

Theorem 2.2. *For any triangle-free graph $G \in \mathfrak{N}_c$ with $\Delta(G) \geq 3$, $W_c(G) \leq |V(G)| + \Delta(G) - 3$.*

Proof. Suppose, for a contradiction, that the theorem is false and let $G \in \mathfrak{N}$ be graph with $\Delta(G) \geq 3$ which does not satisfy the conclusion of the theorem. Let φ be a cyclic interval t -coloring of G satisfying that $t > |V(G)| + \Delta(G) - 3$. If t and 1 are considered to be consecutive colors, then the first color of a set of consecutive colors can be chosen from the set $\{1, \dots, t\}$ in precisely t ways. Let

$$C_\varphi = \{c : c \text{ is the first color of some set } S(v, \varphi) \text{ for a vertex } v \text{ of } G\}.$$

Since $t > |V(G)|$, there is some color i satisfying that $i \notin C_\varphi$. Moreover, by rotating the colors in φ modulo t , we can construct a cyclic interval coloring φ' such that $t \notin C_{\varphi'}$. Now, by removing all edges from G with colors $1, \dots, \Delta(G) - 2$ under φ' , we obtain a graph H such that the restriction of φ' to H is an interval $(t - (\Delta(G) - 2))$ -coloring. Thus, by Theorem 1.2, we have that $t - (\Delta(G) - 2) \leq |V(G)| - 1$, which is a contradiction and the result follows. \square

Part (ii) of Theorem 2.1 implies that Conjecture 1.1 (ii) holds for graphs with maximum degree 2. Proceeding as in the proof of Theorem 2.2, and using Theorem 1.4 instead of Theorem 1.2, it is straightforward to prove the following improvement of that result.

Theorem 2.3. *For any graph $G \in \mathfrak{N}_c$ with $\Delta(G) \geq 3$, $W_c(G) \leq 2|V(G)| + \Delta(G) - 6$.*

Theorems 2.2 and 2.3 imply that Conjecture 1.1 holds for any graph with maximum degree at most 3. Next, we prove that Conjecture 1.1 (i) holds for any graph with maximum degree at most 4.

Theorem 2.4. *For any triangle-free graph $G \in \mathfrak{N}_c$ with $\Delta(G) \leq 4$, $W_c(G) \leq |V(G)|$.*

Proof. Suppose that the theorem is false and let G be a vertex-minimal counterexample with maximum degree 4. Let φ be a cyclic interval t -coloring of G , where $t > |V(G)|$.

Since $t > |V(G)|$, φ is not an interval coloring. Moreover, by a counting argument similar to the one in the proof of the preceding theorems we may assume that at most two vertices

in G are *cyclic* vertices under φ , that is, they are incident to one edge colored 1 and one edge colored t .

Suppose first that only one vertex v in G is cyclic under φ . From G we define a new graph H by splitting v into two new vertices v' and v'' where

- v' is adjacent to all vertices x such that $xv \in E(G)$ and $\varphi(xv) \in \{t-2, t-1, t\}$;
- v'' is adjacent to all vertices x such that $xv \in E(G)$ and $\varphi(xv) \in \{1, 2, 3\}$.

The edge coloring φ induces an edge coloring φ_H of H , and since v is the only cyclic vertex of G , φ_H is an interval t -coloring. Hence, by Theorem 1.2, $t \leq |V(H)| - 1 = |V(G)|$, a contradiction.

Suppose now instead that two vertices $u, v \in V(G)$ are cyclic. We first consider the case when u and v are adjacent. We shall assume that $\varphi(uv) \in \{1, 2, 3\}$ (the case when $\varphi(uv) \in \{t-2, t-1, t\}$ can be done analogously).

From G we define a new graph H by splitting the vertex v into two new vertices v' and v'' , and the vertex u into two new vertices u' and u'' where

- v' is adjacent to all vertices $x \neq u$ such that $xv \in E(G)$ and $\varphi(xv) \in \{t-2, t-1, t\}$;
- v'' is adjacent to all vertices $x \neq u$ such that $xv \in E(G)$ and $\varphi(xv) \in \{1, 2, 3\}$;
- u' is adjacent to all vertices $x \neq v$ such that $xu \in E(G)$ and $\varphi(xu) \in \{t-2, t-1, t\}$;
- u'' is adjacent to all vertices $x \neq v$ such that $xu \in E(G)$ and $\varphi(xu) \in \{1, 2, 3\}$.

Let φ_H be the edge coloring of H induced by φ . Let $J = H + \{u'v', u''v''\}$ and color the edge $u'v'$ with color $t+1$ and $u''v''$ with color $\varphi(uv)$. Denote the obtained coloring of J by φ_J . Since u and v are the only cyclic vertices of G , φ_J is an interval $(t+1)$ -coloring. Consequently, $t+1 \leq |V(J)| - 1 = |V(G)| + 1$, which contradicts our assumption, and the result follows.

Let us now consider the case when u and v are not adjacent. We first prove that we may assume that $S_G(v, \varphi) = S_G(u, \varphi) = \{t-1, t, 1, 2\}$. Suppose, for example, that $t-1$ does not appear at u . Let e be the edge incident with u colored t and set $G' = G - e$. Then the restriction of φ to G' is a cyclic interval coloring with t colors and $|V(G')| = |V(G)|$. Moreover, in the coloring of G' there is a only one cyclic vertex, and we may thus proceed as above. A similar argument applies if color 2 does not appear at one of the vertices u, v . We thus conclude that we may assume that $S_G(v, \varphi) = S_G(u, \varphi) = \{t-1, t, 1, 2\}$.

As above, we form a new graph H from G by splitting the vertex u into two new vertices u' and u'' , and v into two vertices v' and v'' , where

- u' (v') is adjacent to every neighbor of u (v) in G that is joined to u (v) by an edge with color in $\{t-2, t-1, t\}$;
- u'' (v'') is adjacent to every neighbor of u (v) in G that is joined to u (v) by an edge with color in $\{1, 2, 3\}$.

Let φ_H be the interval t -coloring of H induced by φ . Moreover, let x' and y' be the neighbors of u' and v' , respectively, that are colored t under φ_H , and x'' and y'' be the neighbors of u'' and v'' , respectively, that are colored 1 under φ_H . Note that all these vertices are distinct, since u and v are the only cyclic vertices in G .

We first consider the case when $x'y' \in E(H)$ or $x''y'' \in E(H)$. Suppose e.g. that $x'y' \in E(H)$. Then, since H is triangle-free and only colors $t - 1$ and t appear at u' and v' , u' and v' has no common neighbor in H . Hence, the graph $H + u'v'$ is triangle-free and by coloring $u'v'$ with color $t + 1$, we obtain an interval $(t + 1)$ -coloring of $H + u'v'$. Thus $t + 1 \leq |V(H)| - 1 = |V(G)| + 1$, a contradiction. The case when $x''y'' \in E(H)$ is analogous.

Suppose now that $x'y', x''y'' \notin E(H)$. If u' and v' have no common neighbor in H , then we proceed as in the preceding paragraph, so assume that u' and v' have a common neighbor. This means that either $x' \in N_H(u') \cap N_H(v')$ or $y' \in N_H(u') \cap N_H(v')$. Suppose, for instance, that the former holds. We first consider the case when $u'y' \notin E(H)$.

If $N_H(x') \cap N_H(y') = \{v'\}$, then neither $x'v'$ or $v'y'$ is contained in a cycle of length at most 4, because $d_H(v') = 2$. Thus the graph J obtained from $H - v'y'$ by identifying v' and y' is triangle-free. Moreover, the coloring φ_H induces an interval t -coloring of J , and, consequently, $t \leq |V(J)| - 1 = |V(G)|$, which is a contradiction.

If $|N_H(x') \cap N_H(y')| = 2$, then we proceed as follows. If there is an edge colored t under φ_H that is not incident with u' or v' , then we form a graph J by removing the edge $u'x'$ from $H - v'$. Note that the restriction of φ_H to J is an interval t -coloring of J . Hence $t \leq |V(J)| - 1 = |V(G)|$, a contradiction, as above. If all edges colored t under φ_H are incident with u' or v' , then the restriction of φ_H to the graph $H - \{u', v'\}$ is an interval $(t - 1)$ -coloring, unless $d_H(y') = 1$. If $d_H(y') > 1$, then it follows that $t - 1 \leq |V(H)| - 3$, again a contradiction to our above assumption. If $d_H(y') = 1$, then a similar contradiction can be derived by considering the graph $H - \{u', v', y'\}$ and the restriction of φ_H to this graph.

Now we consider the case when $u'y' \in E(H)$. Suppose first that there is some vertex $w \in V(H) \setminus \{u', v', x', y'\}$ where color $t - 1$ appears under φ_H . Then the restriction of φ_H to the graph $H - \{u', v'\}$ is an interval t' -coloring, for some $t' \geq t - 1$, and, consequently, $t - 1 \leq |V(H)| - 2 - 1 = |V(G)| - 1$, a contradiction.

Now assume that there is no vertex $w \in V(H) \setminus \{u', v', x', y'\}$ where color $t - 1$ appears under φ_H . Then u and v are the only vertices in G where colors $t - 1$ and t appear under φ . Denote by x and y respectively the vertices in G such that $\varphi(ux) = t$ and $\varphi(vy) = t$. Consider the graph J obtained from $G - \{ux, uy, vx, vy\}$ by identifying u and x , and v and y , respectively. J is triangle-free and the coloring φ induces an cyclic interval $(t - 2)$ -coloring of J . Hence, J is a counterexample with a smaller number of vertices than G , a contradiction to the choice of G . \square

In the following we shall derive some upper bounds on $W_c(G)$ related to Conjecture 1.1. To this end, we first introduce some new notation. For brevity, a graph G is called an (n, m) -graph if it contains n vertices and m edges. Let G be an (n, m) -graph and $G \in \mathfrak{N}_c$. Also, let α be a cyclic interval $W_c(G)$ -coloring of G and U be the set of all vertices $v \in V(G)$ such that $S(v, \alpha)$ is not an interval of integers. If $U = \emptyset$, then $G \in \mathfrak{N}$ and $W_c(G) \leq W(G) \leq 2|V(G)| - 3$, by Theorem 1.3. Assume that $U \neq \emptyset$. Clearly, for each $u \in U$, there exists a color c_u ($1 < c_u <$

$W_c(G)$) such that $c_u \notin S(u, \alpha)$. For each $u \in U$, we split the neighbors of u into two disjoint sets as follows: $N_G(u) = N_\alpha^<(u) \cup N_\alpha^>(u)$, where $N_\alpha^<(u) = \{v \in N_G(u) : 1 \leq \alpha(uv) < c_u\}$ and $N_\alpha^>(u) = \{v \in N_G(u) : c_u < \alpha(uv) \leq W_c(G)\}$. (Note that these sets are uniquely determined from the cyclic interval coloring α .) We construct a new graph $S_{\alpha,U}(G)$ by splitting all the vertices of U as follows: for each $u \in U$, we delete u from G and add two new vertices $u^<$ and $u^>$; then we join the vertex $u^<$ with the vertices $N_\alpha^<(u)$ and color the edge $u^<v$ by the color $\alpha(uv)$ for every $v \in N_\alpha^<(u)$; next we join the vertex $u^>$ with the vertices $N_\alpha^>(u)$ and color the edge $u^>v$ by the color $\alpha(uv)$ for every $v \in N_\alpha^>(u)$; finally we color all the remaining edges using the same colors as in G . Clearly, $S_{\alpha,U}(G)$ is an $(n + |U|, m)$ -graph and $S_{\alpha,U}(G) \in \mathfrak{N}$.

Lemma 2.5. (*Translation Lemma*) Let $\mathfrak{C} \subseteq \mathfrak{N}_c$, $\mathfrak{C}' \subseteq \mathfrak{N}$ and assume that $W(G') \leq f(|V(G')|)$ holds for any graph $G' \in \mathfrak{C}'$, where f is a monotonically non-decreasing function. If for every $G \in \mathfrak{C}$, there is a cyclic interval $W_c(G)$ -coloring α of G and a subset $U \subseteq V(G)$, such that the graph $S_{\alpha,U}(G)$ belongs to \mathfrak{C}' , then for any $G \in \mathfrak{C}$, we have

$$W_c(G) \leq f\left(|V(G)| + \left\lfloor \frac{2|E(G)| - |V(G)|}{W_c(G)} \right\rfloor\right).$$

Proof. Let G be an (n, m) -graph and $G \in \mathfrak{C}$. Also, let $t = W_c(G)$. Consider a cyclic interval t -coloring α of G . We say the color i ($1 \leq i \leq t$) *splits* the vertex $v \in V(G)$ if $\{i-1, i\} \subseteq S(v, \alpha)$ (modulo t). For each $v \in V(G)$, let $S'(v, \alpha)$ denote the set of all colors i that splits the vertex v . Clearly, $|S'(v, \alpha)| = d_G(v) - 1$ for every $v \in V(G)$. By the pigeonhole principle, there exists a color that splits no more than $\left\lfloor \frac{\sum_{v \in V(G)} (d_G(v) - 1)}{t} \right\rfloor = \left\lfloor \frac{2m - n}{t} \right\rfloor$ vertices. Without loss of generality we may assume that this color is 1 (by rotating the colors in α modulo t , we can construct such a coloring). Let U be the set of vertices of G that are splitted by color 1. Then $|U| \leq \left\lfloor \frac{2m - n}{t} \right\rfloor$. Let us now consider the graph $S_{\alpha,U}(G)$ defined above. If $S_{\alpha,U}(G) \in \mathfrak{C}'$, then

$$W_c(G) = t \leq W(S_{\alpha,U}(G)) \leq f(|V(S_{\alpha,U}(G))|) \leq f\left(n + \left\lfloor \frac{2m - n}{t} \right\rfloor\right).$$

□

The following two theorems are improvements of the bounds on $W_c(G)$ in Theorem 2.1 for graphs with large maximum degree.

Theorem 2.6. If G is a triangle-free graph and $G \in \mathfrak{N}_c$, then

$$W_c(G) \leq \frac{\sqrt{3}+1}{2}(|V(G)| - 1).$$

Proof. Let G be a triangle-free (n, m) -graph and \mathfrak{C} (\mathfrak{C}') be the set of all cyclically interval colorable (interval colorable) triangle-free graphs. By Theorem 1.2, we have $W(G') \leq |V(G')| - 1$ for every $G' \in \mathfrak{C}'$.

Let $t = W_c(G)$ and consider a cyclic interval t -coloring α of G . As in the proof of Lemma 2.5, we construct a graph $S_{\alpha,U}(G)$ from the coloring α , G and U , where $|U| \leq \left\lfloor \frac{2m - n}{t} \right\rfloor$. Since $S_{\alpha,U}(G) \in \mathfrak{C}'$, by Lemma 2.5, we obtain

$$t \leq W(S_{\alpha,U}(G)) \leq |V(S_{\alpha,U}(G))| - 1 \leq n + \left\lfloor \frac{2m - n}{t} \right\rfloor - 1.$$

On the other hand, since G is triangle-free, by Mantel's theorem, we have $2m \leq \frac{n^2}{2}$. Taking this into account, we obtain

$$t \leq n + \left\lfloor \frac{2m-n}{t} \right\rfloor - 1 \leq n + \frac{\frac{n^2}{2}-n}{t} - 1.$$

This yields a quadratic inequality $t^2 - (n-1)t - n(\frac{n}{2} - 1) \leq 0$. Thus,

$$W_c(G) = t \leq \frac{1}{2} (n-1 + \sqrt{3n^2 - 6n + 1}) \leq \frac{\sqrt{3}+1}{2} (|V(G)| - 1).$$

□

Theorem 2.7. *If G is a graph with at least two vertices and $G \in \mathfrak{N}_c$, then*

$$W_c(G) \leq (\sqrt{3} + 1)|V(G)| - 3.$$

Proof. Let G be a (n, m) -graph ($n \geq 2$) and \mathfrak{C} (\mathfrak{C}') be the set of all cyclically interval colorable (interval colorable) graphs with at least two vertices. By Theorem 1.3, we have $W(G') \leq 2|V(G')| - 3$ for every $G' \in \mathfrak{C}'$.

Set $t = W_c(G)$ and consider a cyclic interval t -coloring α of G . As in the proof of Lemma 2.5, we construct a graph $S_{\alpha, U}(G)$ from the coloring α , G and U , where $|U| \leq \left\lfloor \frac{2m-n}{t} \right\rfloor$. Since $S_{\alpha, U}(G)$ belongs to \mathfrak{C}' , we deduce from Lemma 2.5 that

$$t \leq W(S_{\alpha, U}(G)) \leq 2|V(S_{\alpha, U}(G))| - 3 \leq 2(n + \left\lfloor \frac{2m-n}{t} \right\rfloor) - 3.$$

On the other hand, since $2m \leq n(n-1)$, we obtain

$$t \leq 2(n + \left\lfloor \frac{2m-n}{t} \right\rfloor) - 3 \leq 2(n + \frac{n(n-1)-n}{t}) - 3.$$

This yields a quadratic inequality $t^2 - (2n-3)t - 2n(n-2) \leq 0$. Thus,

$$W_c(G) = t \leq \frac{1}{2} (2n-3 + \sqrt{12n^2 - 28n + 9}) \leq (\sqrt{3} + 1)|V(G)| - 3.$$

□

Our next result is an improvement of part (ii) of Theorem 2.1 for graphs G with average degree $d(G) < \Delta(G)$.

Theorem 2.8. *If G is a graph with at least three vertices and $G \in \mathfrak{N}_c$, then*

$$W_c(G) \leq 2|V(G)| + d(G) - 3.5.$$

Proof. Let G be an (n, m) -graph ($n \geq 3$) and \mathfrak{C} (\mathfrak{C}') be the set of all cyclically interval colorable (interval colorable) graphs with at least two vertices. Also, let $t = W_c(G)$ and $d = d(G)$. By proceeding as above, we consider a cyclic interval t -coloring α of G and deduce, using Lemma 2.5, that

$$t \leq W(S_{\alpha, U}(G)) \leq 2|V(S_{\alpha, U}(G))| - 3 \leq 2(n + \left\lfloor \frac{2m-n}{t} \right\rfloor) - 3.$$

On the other hand, since $2m \leq dn$, we obtain

$$t \leq 2(n + \left\lfloor \frac{2m-n}{t} \right\rfloor) - 3 \leq 2(n + \frac{dn-n}{t}) - 3.$$

As above, this yields a quadratic inequality $t^2 - (2n-3)t - 2n(d-1) \leq 0$. Thus,

$$W_c(G) = t \leq \frac{1}{2} \left(2n - 3 + \sqrt{4n^2 - (20 - 8d)n + 9} \right) \leq 2|V(G)| + d(G) - 3.5.$$

□

Since any planar graph has average degree less than 6, we deduce the following from the preceding theorem.

Corollary 2.9. *If G is a planar graph and $G \in \mathfrak{N}_c$, then*

$$W_c(G) \leq 2|V(G)| + 2.$$

For triangle-free sparse graphs G with a cyclic interval coloring we can prove a better upper bound on $W_c(G)$.

Theorem 2.10. *Let $G \in \mathfrak{N}_c$ be a triangle-free (n, m) -graph, and a and b positive numbers satisfying $m \leq a \cdot n + b$. Then if $8b + 1 \leq (3 - 4a)^2$, then*

$$W_c(G) \leq |V(G)| + 2a - 2.$$

Proof. Let G be a triangle-free (n, m) -graph ($m \leq a \cdot n + b$) and \mathfrak{C} (\mathfrak{C}') be the set of all cyclically interval colorable (interval colorable) triangle-free graphs. Also, let $t = W_c(G)$. Consider a cyclic interval t -coloring α of G . As in the proof of Theorem 2.6, we have that

$$t \leq n + \left\lfloor \frac{2m-n}{t} \right\rfloor - 1.$$

On the other hand, since $m \leq a \cdot n + b$, we obtain

$$t \leq n + \left\lfloor \frac{2m-n}{t} \right\rfloor - 1 \leq n + \frac{2an+2b-n}{t} - 1.$$

This yields a quadratic inequality $t^2 - (n-1)t - (n(2a-1) + 2b) \leq 0$, implying that

$$W_c(G) = t \leq \frac{1}{2} \left(n - 1 + \sqrt{n^2 - 2n(3-4a) + 8b + 1} \right).$$

Now taking into account that $8b + 1 \leq (3 - 4a)^2$, we obtain

$$W_c(G) \leq \frac{1}{2} \left(n - 1 + \sqrt{n^2 - 2n(3-4a) + 8b + 1} \right) \leq |V(G)| + 2a - 2.$$

□

Corollary 2.11. *If G is a triangle-free planar graph and $G \in \mathfrak{N}_c$, then*

$$W_c(G) \leq |V(G)| + 2.$$

Corollary 2.12. *If G is a triangle-free outerplanar graph with at least two vertices and $G \in \mathfrak{N}_c$, then*

$$W_c(G) \leq |V(G)| + 1.$$

We note that the last two results are almost sharp since $W_c(C_n) = n$, where C_n is a cycle on n vertices.

3 Bounds on $W(G)$

In this section we prove some bounds on $W(G)$ for interval colorable multigraphs G . Let us first recall some other bounds on $W(G)$ that appear in the literature.

The first lower bound on $W(G)$ for interval colorable regular graphs G was obtained by Kamalian [16]. In particular, he proved that if G is a regular graph with $\chi'(G) = \Delta(G)$ and $3 \leq |V(G)| \leq 2^{\Delta(G)} + 1$, then $W(G) \geq \Delta(G) + \lfloor \log_2(|V(G)| - 1) \rfloor$. In [6], Axenovich proved that if G is a planar graph and $G \in \mathfrak{N}$, then $W(G) \leq \frac{11}{6}|V(G)|$, and conjectured that for all interval colorable planar graphs, $W(G) \leq \frac{3}{2}|V(G)|$. In [5], Asratian and Kamalian proved that if G is connected and $G \in \mathfrak{N}$, then

$$W(G) \leq (\text{diam}(G) + 1)(\Delta(G) - 1) + 1.$$

If we, in addition, assume that G is bipartite, then this bound can be improved to $W(G) \leq \text{diam}(G)(\Delta(G) - 1) + 1$. Recently, Kamalian and Petrosyan [17] showed that these upper bounds cannot be significantly improved. In this section we shall derive a similar upper bound on $W(G)$ for interval colorable multigraphs G based on the circumference of G . First we give a short proof of Theorem 1.3 based on Theorem 1.2; the original proof by Kamalian is rather lengthy.

Proof of Theorem 1.3. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ and α be an interval $W(G)$ -coloring of the graph G . Define an auxiliary graph H as follows:

$$V(H) = U \cup W, \text{ where}$$

$$U = \{u_1, u_2, \dots, u_n\}, W = \{w_1, w_2, \dots, w_n\} \text{ and}$$

$$E(H) = \{u_i w_j, u_j w_i : v_i v_j \in E(G), 1 \leq i \leq n, 1 \leq j \leq n\} \cup \{u_i w_i : 1 \leq i \leq n\}.$$

Clearly, H is a bipartite graph with $|V(H)| = 2|V(G)|$.

Define an edge-coloring β of H as follows:

- (1) for every edge $v_i v_j \in E(G)$, let $\beta(u_i w_j) = \beta(u_j w_i) = \alpha(v_i v_j) + 1$,
- (2) for $i = 1, 2, \dots, n$, let $\beta(u_i w_i) = \overline{S}(v_i, \alpha) + 2$.

It is easy to see that β is an edge-coloring of the graph H with colors $2, 3, \dots, W(G) + 2$ and $\underline{S}(u_i, \beta) = \underline{S}(w_i, \beta)$ for $i = 1, 2, \dots, n$. Now we present an interval $(W(G) + 2)$ -coloring of the graph H . For that we take one edge $u_{i_0} w_{i_0}$ with $\underline{S}(u_{i_0}, \beta) = \underline{S}(w_{i_0}, \beta) = 2$, and recolor it with color 1. Clearly, such a coloring is an interval $(W(G) + 2)$ -coloring of H . Since H is a bipartite graph and $H \in \mathfrak{N}$, by Theorem 1.2, we have

$$W(G) + 2 \leq |V(H)| - 1 = 2|V(G)| - 1, \text{ thus}$$

$$W(G) \leq 2|V(G)| - 3.$$

□

Theorem 3.1. *If G is a 2-connected multigraph and $G \in \mathfrak{N}$, then*

$$W(G) \leq 1 + \left\lfloor \frac{c(G)}{2} \right\rfloor (\Delta(G) - 1).$$

Proof. Consider an interval $W(G)$ -coloring α of G . In the coloring α of G , we consider the edges with colors 1 and $W(G)$. Let $e = uv$, $e' = u'v'$ and $\alpha(e) = 1$, $\alpha(e') = W(G)$. Since G is 2-connected, there is a cycle C that contains both edges e and e' . Clearly, $|V(C)| \leq c(G)$. We label the vertices of C in two directions: from u to u' and from v to v' . Let $P = u_1, \dots, u_s$ and $Q = v_1, \dots, v_t$ be two paths on C from u to u' and from v to v' , respectively, where $u_1 = u, u_s = u'$ and $v_1 = v, v_t = v'$ ($s, t \geq 1$). Clearly, $\min\{s, t\} \leq \left\lfloor \frac{c(G)}{2} \right\rfloor$. Without loss of generality we may assume that $s \leq \left\lfloor \frac{c(G)}{2} \right\rfloor$.

Since α is an interval $W(G)$ -coloring of G , we have

$$\begin{aligned} \alpha(u_1 u_2) &\leq d_G(u_1), \\ \alpha(u_2 u_3) &\leq \alpha(u_1 u_2) + d_G(u_2) - 1, \\ &\dots\dots\dots \\ \alpha(u_i u_{i+1}) &\leq \alpha(u_{i-1} u_i) + d_G(u_i) - 1, \\ &\dots\dots\dots \\ \alpha(u_{s-1} u_s) &\leq \alpha(u_{s-2} u_{s-1}) + d_G(u_{s-1}) - 1, \\ W(G) = \alpha(e') &= \alpha(u'v') \leq \alpha(u_{s-1} u_s) + d_G(u_s) - 1. \end{aligned}$$

Summing up these inequalities, we obtain

$$W(G) \leq 1 + \sum_{i=1}^s (d_G(u_i) - 1) \leq 1 + \left\lfloor \frac{c(G)}{2} \right\rfloor (\Delta(G) - 1).$$

□

Corollary 3.2. *If G is a 2-connected multigraph and $G \in \mathfrak{N}$, then*

$$W(G) \leq 1 + \left\lfloor \frac{|V(G)|}{2} \right\rfloor (\Delta(G) - 1).$$

Corollary 3.3. *If G is a 2-connected multigraph with $\Delta(G) \leq 4$ and $G \in \mathfrak{N}$, then*

$$W(G) \leq 3 \left\lfloor \frac{|V(G)|}{2} \right\rfloor + 1.$$

Corollary 3.4. *If G is a 2-connected planar graph with $\Delta(G) \leq 4$ and $G \in \mathfrak{N}$, then*

$$W(G) \leq \frac{3}{2}|V(G)|.$$

Proof. First let us note that Corollary 3.3 implies that the statement is true for planar graphs with an odd number of vertices. Moreover, we have $W(G) \leq \frac{3}{2}|V(G)| + 1$. Assume that $|V(G)| = 2n$ ($n \in \mathbb{N}$). Consider an interval $(3n+1)$ -coloring α of G . As in the proof of Theorem 3.1, we consider the edges with colors 1 and $3n+1$. Let $e = uv$, $e' = u'v'$ and $\alpha(e) = 1$, $\alpha(e') = 3n+1$. Since G is 2-connected, there is a cycle C that contains both edges e and e' . Clearly, $|V(C)| \leq 2n$. We label the vertices of C in two directions: from u to u' and from v to v' . Let $P = u_1, \dots, u_s$ and $Q = v_1, \dots, v_t$ be two paths on C from u to u' and from v to v' , respectively, where $u_1 = u, u_s = u'$ and $v_1 = v, v_t = v'$ ($s, t \geq 1$). It is easy to see that $s = t = \frac{|V(G)|}{2} = n$ (otherwise, by considering the shortest path, we obtain that $W(G) \leq 3n$). Moreover, we have

$$\begin{aligned}
\alpha(u_1u_2) &= \alpha(v_1v_2) = 4, \\
\alpha(u_2u_3) &= \alpha(v_2v_3) = 7, \\
&\dots\dots\dots \\
\alpha(u_iu_{i+1}) &= \alpha(v_iu_{i+1}) = 3i + 1, \\
&\dots\dots\dots \\
\alpha(u_{s-1}u_s) &= \alpha(v_{t-1}v_t) = 3n - 2, \\
\alpha(e') &= \alpha(u'v') = \alpha(u_sv_t) = \alpha(u_nv_n) = 3n + 1.
\end{aligned}$$

This implies that $\overline{S}(u_i, \alpha) = \overline{S}(v_i, \alpha) = 3i + 1$ for $i = 1, \dots, n$. Let us consider the edges with color $3n$. These edges can only be incident to vertices u_n and v_n , since $\overline{S}(w, \alpha) < 3n$ for any $w \in V(G) \setminus \{u_n, v_n\}$. This implies that the edges with colors $3n$ and $3n + 1$ must be parallel, which is a contradiction. \square

The last corollary above shows that Axenovich's conjecture [6] is true for all interval colorable 2-connected planar graphs with maximum degree at most 4.

It is well-known that if a connected regular multigraph G has a cut-vertex, then $\chi'(G) > \Delta(G)$. Moreover, by Proposition 1.5, if G is interval colorable, then $\chi'(G) = \Delta(G)$. Thus, if a connected regular multigraph is interval colorable, then it is 2-connected. Hence, we have the following two results.

Corollary 3.5. *If G is a connected r -regular multigraph and $G \in \mathfrak{N}$, then*

$$W(G) \leq 1 + \left\lfloor \frac{|V(G)|}{2} \right\rfloor (r - 1).$$

Corollary 3.6. *If G is a connected cubic multigraph and $G \in \mathfrak{N}$, then*

$$W(G) \leq |V(G)| + 1.$$

We now show that all aforementioned upper bounds are sharp.

Proposition 3.7. *For any integers $n, r \geq 2$, there exists a 2-connected multigraph G with $|V(G)| = 2n$ and $\Delta(G) = r$ such that $G \in \mathfrak{N}$ and $W(G) = 1 + n(r - 1)$.*

Proof. For the proof, we are going to construct a multigraph $G_{n,r}$ that satisfies the specified conditions. We define the multigraph $G_{n,r}$ ($n, r \geq 2$) as follows:

- 1) $V(G_{n,r}) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$,
- 2) $E(G_{n,r})$ contains $r - 1$ parallel edges between the vertices u_1 and v_1 , and $r - 1$ parallel edges between u_n and v_n , for $2 \leq i \leq n - 1$, u_i and v_i are joined by $r - 2$ parallel edges, and for $1 \leq j \leq n - 1$, $E(G_{n,r})$ contains the edges u_ju_{j+1}, v_jv_{j+1} .

$G_{n,r}$ is a 2-connected multigraph with $|V(G_{n,r})| = 2n$ and $\Delta(G_{n,r}) = r$.

Let us show that $G_{n,r}$ has an interval $(1 + n(r - 1))$ -coloring. We define an edge-coloring α of $G_{n,r}$ as follows: first we color the edges from $E(u_1v_1)$ with colors $1, \dots, r - 1$ and from $E(u_nv_n)$ with colors $(n - 1)(r - 1) + 2, \dots, n(r - 1) + 1$; then we color the edges from $E(u_iv_i)$ with colors $(i - 1)(r - 1) + 2, \dots, i(r - 1)$, where $2 \leq i \leq n - 1$; finally we color the edges u_ju_{j+1} and v_jv_{j+1} with color $j(r - 1) + 1$, where $1 \leq j \leq n - 1$. It is straightforward to verify that α is an interval $(1 + n(r - 1))$ -coloring of $G_{n,r}$. Thus, $G_{n,r} \in \mathfrak{N}$ and $W(G_{n,r}) \geq 1 + n(r - 1)$. On the other hand, by Corollary 3.2, we have $W(G_{n,r}) \leq 1 + n(r - 1)$, so $W(G_{n,r}) = 1 + n(r - 1)$. \square

4 Bounds on $w(G)$

In this section we prove some bounds on the minimum number of colors in an interval coloring of a multigraph G . Hanson and Loten [13] proved a lower bound on the number of colors in an interval coloring of an (a, b) -biregular graph. Here we give a similar bound.

Theorem 4.1. *If G is a multigraph and $G \in \mathfrak{N}$, then*

$$w(G) \geq \left\lceil \frac{|V(G)|}{2\alpha'(G)} \right\rceil \delta(G).$$

Proof. Let $\delta = \delta(G)$. Consider an interval $w(G)$ -coloring α of G . In the coloring α of G , we consider the edges with colors $\delta, 2\delta, \dots, k\delta$, where k is the maximum integer for which there exists an edge with color $k\delta$. Clearly, $k\delta \leq w(G)$. Since each vertex v of G is incident to at least one of the edges with color $\delta, 2\delta, \dots, k\delta$, we have

$$|V(G)| \leq \sum_{i=1}^k 2|M_\alpha(i \cdot \delta)| \leq 2k \cdot \alpha'(G),$$

where $M_\alpha(i \cdot \delta) = \{e \in E(G) : \alpha(e) = i \cdot \delta\}$, $1 \leq i \leq k$.

This implies that $k \geq \left\lceil \frac{|V(G)|}{2\alpha'(G)} \right\rceil$. On the other hand, since $k\delta \leq w(G)$, we obtain $w(G) \geq \left\lceil \frac{|V(G)|}{2\alpha'(G)} \right\rceil \delta(G)$. \square

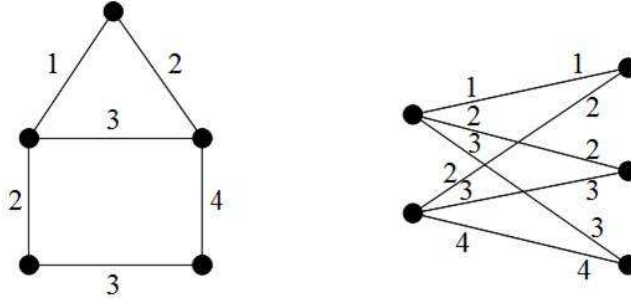


Figure 1: Graphs with maximum degree 3 without an interval 3-coloring.

The graphs in Figure 1 show that the lower bound in Theorem 4.1 is sharp. Note also that there are infinite families of graphs, such as all $(a-1, a)$ -biregular graphs, for which the lower bound in Theorem 4.1 is sharp.

Corollary 4.2. *If a multigraph G has no perfect matching and $G \in \mathfrak{N}$, then*

$$w(G) \geq \max\{\Delta(G), 2\delta(G)\}.$$

Using similar counting arguments we can prove that Eulerian multigraphs with an odd number of edges do not have interval colorings. We first prove a more general theorem.

Theorem 4.3. *If for a multigraph G , there exists a number d such that d divides $d_G(v)$ for every $v \in V(G)$ and d does not divide $|E(G)|$, then $G \notin \mathfrak{N}$.*

Proof. Suppose, to the contrary, that G has an interval t -coloring α for some $t \geq \Delta(G)$. Let d be a number such that d divides $d_G(v)$ for every $v \in V(G)$. We call an edge $e \in E(G)$ a d -edge if $\alpha(e) = d \cdot l$ for some $l \in \mathbb{N}$. Since α is an interval coloring, we have that for any $v \in V(G)$, the set $S(v, \alpha)$ contains exactly $\frac{d_G(v)}{d}$ d -edges. Now let m_d be the number of d -edges in G . By the Handshaking lemma, we obtain $m_d = \frac{1}{2} \sum_{v \in V(G)} \frac{d_G(v)}{d} = \frac{|E(G)|}{d}$. Hence, d divides $|E(G)|$, which is a contradiction. \square

Corollary 4.4. *If G is an Eulerian multigraph and $|E(G)|$ is odd, then $G \notin \mathfrak{N}$.*

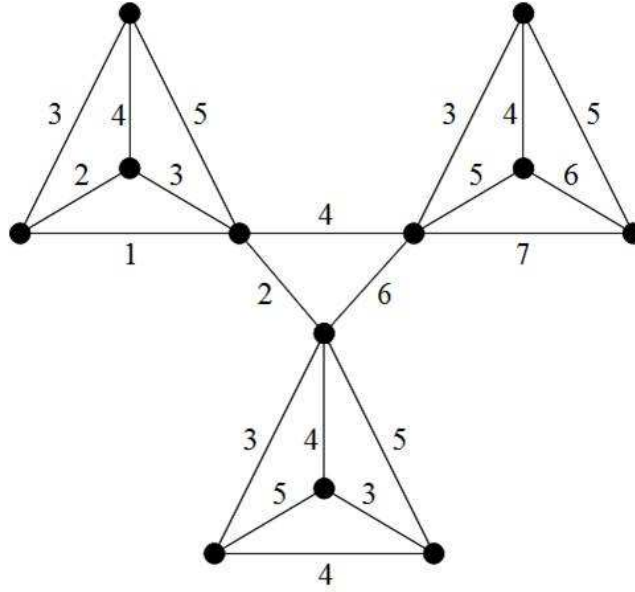


Figure 2: An interval colorable connected odd graph G with $|E(G)| - \frac{|V(G)|}{2} = 15$ and $w(G) \geq 6$.

Finally, we prove an analogue of Corollary 4.2 for odd multigraphs.

Theorem 4.5. *If G is an odd multigraph, $|E(G)| - \frac{|V(G)|}{2}$ is odd and $G \in \mathfrak{N}$, then*

$$w(G) \geq \max\{\Delta(G), 2\delta(G)\}.$$

Proof. Let $\delta = \delta(G)$. If G has no perfect matching, then the result follows from Corollary 4.2. Assume that G has a perfect matching. Next suppose, to the contrary, that G has an interval t -coloring α for some $t \leq 2\delta - 1$. Since for every $v \in V(G)$, $1 \leq \underline{S}(v, \alpha) \leq \delta$, we obtain that $\delta \in \bigcap_{v \in V(G)} S(v, \alpha)$. This implies that the edges with color δ form a perfect matching in G . Let M be this perfect matching. Consider the multigraph $G - M$. We define an edge-coloring β of $G - M$ as follows: for every $e \in E(G - M)$, let

$$\beta(e) = \begin{cases} \alpha(e), & \text{if } 1 \leq \alpha(e) \leq \delta - 1, \\ \alpha(e) - 1, & \text{if } \delta + 1 \leq \alpha(e) \leq t. \end{cases}$$

It is not difficult to see that β is an interval $(t-1)$ -coloring of $G-M$. Since $|E(G)| - \frac{|V(G)|}{2}$ is odd, we obtain that $G-M$ is an even multigraph with an odd number of edges. This implies that $G-M$ has an Eulerian component with an odd number of edges which is interval colorable, but this contradicts Corollary 4.4. \square

The graph in Figure 2 shows that the lower bound in the preceding theorem is sharp.

As we have seen, there are several lower bounds on $w(G)$ for different families of interval colorable graphs. In particular, since a complete bipartite graph $K_{a,b}$ requires at least $a + b - \gcd(a, b)$ colors for an interval coloring [15, 16, 12], it holds that for any positive integer d , there is a graph G such that $G \in \mathfrak{N}$ and $w(G) - \chi'(G) \geq d$. It is not known if a similar result holds for cyclic interval colorings. We would like to suggest the following.

Problem 1. *For any positive integer d , is there a graph G such that $G \in \mathfrak{N}_c$ and $w_c(G) - \chi'(G) \geq d$?*

Even the case $d = 1$ of the above problem is open. The problem has a positive answer if $d = 1$ and $\chi'(G) = \Delta(G)$ [22]. Another variant of the problem is obtained by replacing $\chi'(G)$ by $\Delta(G)$. The answer to this latter problem is positive for multigraphs, but it is open for $d \geq 2$ in the case of graphs. For the case $d = 1$, this reformulated problem has a positive answer for ordinary graphs (see e.g. [22, 1]).

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